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Construction of Modular Branching Functions from Bethe's Equations in the 3-State Potts Chain

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Abstract

We use the single particle excitation energies and the completeness rules of the 3-state anti-ferromagnetic Potts chain, which have been obtained from Bethe's equation, to compute the modular invariant partition function. This provides a fermionic construction for the branching functions of the D_4 representation of Z_4 parafermions which complements the previous bosonic constructions. It is found that there are oscillations in some of the correlations and a new connection with the field theory of the Lee-Yang edge is presented.

1 Introduction

The theory of integrable quantum spin chains was initiated by Bethe in 1931 [1] in his study of the spin 1/2 Heisenberg anti-ferromagnet. From that beginning, particularly in the last 20 years, an enormous number of one dimensional quantum spin systems have been discovered which, along with their two dimensional statistical counterparts, have the remarkable property that their energy eigenvalues are given by the solutions of a system of equations which have become known as Bethe's equations:

$$(-1)^{M+1}\left[\frac{\sinh(\lambda_j - iS\gamma)}{\sinh(\lambda_j + iS\gamma)}\right]^N = \prod_{k=1}^{L}\frac{\sinh(\lambda_j-\lambda_k-i\gamma)}{\sinh(\lambda_j-\lambda_k+i\gamma)}. \tag{1.1}$$

Here M is the number of sites in the chain, N and L are related to M (typically $N = M$ or $2M$), and S and γ are parameters which characterize the specific models.

One of the features derived from these Bethe's equations is that in the limit $M \to \infty$ the spectrum of lowlying excitations E_{ex} above the ground state is expressed in terms of a set of

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single particle levels $e_\alpha(P)$ depending on a momentum P and combined with a set of rules as

$$E_{ex} - E_{GS} = \sum_{\alpha, rules} e_\alpha(P_i), \tag{1.2}$$

$$P = \sum_{\alpha, rules} P_i^\alpha, \tag{1.3}$$

and almost without exception one of the rules of combination is a "fermi" exclusion rule:

$$P_i^\alpha \neq P_j^\alpha \quad \text{if} \quad i \neq j. \tag{1.4}$$

The form for energy levels (1.2) and (1.3) is referred to as a quasi-particle spectrum. Furthermore, in many of these spin chains one or more e_α vanish as $P \to 0$

$$e(P) \sim v|P|, \tag{1.5}$$

where v is positive and is called the speed of sound.

Much more recently, in 1984, a powerful new formalism was invented [2] to study those integrable systems for which there is no mass gap and (1.5) holds. This method, known as conformal field theory, is more axiomatic. It deals with a continuum approximation to the spin chain (or two dimensional statistical system) and instead of starting from a Hamiltonian it starts from a symmetry principle such as the Virasoro algebra, a Kac-Moody algebra, or modular invariance which captures many of the essential features that lead to the integrability of the systems described by Bethe's equation (1.1).

One of the main objects of computation in conformal field theory is the partition function which is expressed by means of modular invariance [3, 4] in terms of Virasoro characters or (more generally) branching functions $b_j(q)$ as

$$Z = \sum N_{k,l} b_k(q) b_l(\bar{q}). \tag{1.6}$$

Here $q(\bar{q})$ refers to the right (left) moving excitations with

$$q, \bar{q} = e^{-\frac{2\pi v}{M k_B T}}, \tag{1.7}$$

where T is the temperature and k_B is Boltzmann's constant.

The Virasoro characters and branching functions $b_i(q)$ are solutions to the equations of modular transformation [5–8]. Their construction typically starts with one or more free boson Fock spaces, and then excludes certain null vectors. They are typically given by explicit formulas with several powers of the product

$$Q(q) = \prod_{n=1}^{\infty} (1 - q^n) \tag{1.8}$$

in the denominator, and a power series in q in the numerator, times a fractional power which is usually written as $q^{-c/24+h_k}$. Here the constant c is referred to as the central charge, and h_k are known as the conformal dimensions.

The question now arises as to the relation between the solutions of Bethe's equations and the results of conformal field theory. In particular, one wants to compute the partition function (1.6) starting from Bethe's equations (1.1) (or related functional equations). Recently an important advance in this project was made by Klümper and Pearce [9,10], who computed the central charge and conformal dimensions for the A_{N+1} series of the $A_1^{(1)}$ models classified by Pasquier [11]. However the computation of the full character expansion (1.6) and its relation to the quasi-particle energy spectrum (1.2) is still lacking.

It our purpose here to complete this project and to compute the full partition function (1.6) for a particular quantum spin model: The anti-ferromagnetic 3-state Potts chain. In particular, we will show that the partition function is constructed from the single particle levels of (1.2) which satisfy the fermi exclusion rule (1.4). This provides a physical interpretation of the model, which complements the usual computation that starts with free bosons. The result of conformal field theory, as obtained by Pearce [12], is that the partition function of the spin system is

$$Z = \sum e^{-E_n/k_BT} = e^{-Me_0/k_BT} Z_{pf4}, \tag{1.9}$$

where e_0 is the ground state energy per site [13] and Z_{pf4} is the D_4 representation of the Z_4 parafermionic partition function of [14]:

$$Z_{pf4} = [b_0^0(q) + b_4^0(q)][b_0^0(\bar{q}) + b_4^0(\bar{q})] + 4b_2^0(q)b_2^0(\bar{q}) + 2b_0^2(q)b_0^2(\bar{q}) + 2b_2^2(q)b_2^2(\bar{q}), \tag{1.10}$$

b_m^ℓ given in one of the equivalent bosonic forms of (2.25), (2.27) and appendix B. We here obtain (1.10) starting from Bethe's equation (1.1) for the finite lattice and obtain new fermionic representations for b_m^ℓ given by (3.13) and (3.19) for b_0^0, b_2^0, andb_4^0 in $Q = 0$ and (4.16) for b_0^2 and b_2^2 in $Q = \pm 1$.

Our method is to combine the results of [13], which derives the spectrum of the anti-ferromagnetic 3-state Potts model of the form (1.2), staring from the Bethe's equation derived by Albertini [15], with the completeness study of [16] and the finite size corrections of [10]. In section 2 we summarize the results of these papers which are needed here, as well as the conformal field theory predictions for the model. In Section 3 we compute the partition function in the channel $Q = 0$ and in section 4 we do the same in the channel $Q = 1$. We conclude in section 5 with a discussion of the physical implication of our results in terms of what we call an infrared anomaly. We also discuss the oscillations which are predicted to occur in the correlation functions, and a connection with the field theory [17, 18] of the Lee-Yang edge [19] of the Ising model.

2 Formulation

The 3-state anti-ferromagnetic Potts chain is specified by the Hamiltonian

$$H = \frac{2}{\sqrt{3}} \sum_{j=1}^{M} \{X_j + X_j^\dagger + Z_j Z_{j+1}^\dagger + Z_j^\dagger Z_{j+1}\}, \tag{2.1}$$

where

$$X_j = I \otimes I \otimes \cdots \otimes \underbrace{X}_{j^{th}} \otimes \cdots \otimes I, \tag{2.2}$$

$$Z_j = I \otimes I \otimes \cdots \otimes \underbrace{Z}_{j^{th}} \otimes \cdots \otimes I. \tag{2.3}$$

Here I is the 3×3 identity matrix, the elements of the 3×3 matrices X and Z are

$$X_{j,k} = \delta_{j,k+1} \quad (\mathrm{mod} 3), \tag{2.4}$$

$$Z_{j,k} = \delta_{j,k}\omega^{j-1}, \tag{2.5}$$

$$\omega = e^{2\pi i/3}, \tag{2.6}$$

and we impose periodic boundary conditions $Z_{M+1} \equiv Z_1$.

This spin chain is invariant under translations and under spin rotations. Thus the eigenvalues may by classified in terms of P, the total momentum of the state, and Q, where $e^{2\pi iQ/3}$ is the eigenvalue of the spin rotation operator. Here $P = 2\pi n/M$ where n is an integer $0 \leq n \leq M-1$, and $Q = 0, \pm 1$. Furthermore because H is invariant under complex conjugation there is a conserved C parity of ± 1 in the sector $Q = 0$ and the sectors $Q = \pm 1$ are degenerate.

This spin chain is integrable because of its connection with the integrable 3-state Potts model of statistical mechanics. The eigenvalues satisfy functional equations [15] [20] which are solved in terms of a Bethe's equation (1.1) [15] with

$$N = 2M, \quad \gamma = \pi/3, \quad S = 1/4, \tag{2.7}$$

and

$$L = 2(M - |Q|) \quad \text{for} \quad Q = 0, \pm 1. \tag{2.8}$$

In terms of these λ_k, the eigenvalues of the transfer matrix of the statistical model are

$$\Lambda(\lambda) = \left[\frac{\sinh(\frac{\pi i}{6})\sinh(\frac{\pi i}{3})}{\sinh(\frac{\pi i}{4} - \lambda)\sinh(\frac{\pi i}{4} + \lambda)}\right]^M \prod_{k=1}^{L} \frac{\sinh(\lambda - \lambda_k)}{\sinh(\frac{\pi i}{12} + \lambda_k)}, \tag{2.9}$$

the eigenvalues of the Hamiltonian (2.1) are

$$E = \sum_{k=1}^{L} \cot(i\lambda_k + \frac{\pi}{12}) - \frac{2M}{\sqrt{3}}, \tag{2.10}$$

and the corrsponding momentum is

$$e^{iP} = \Lambda(-i\pi/12) = \prod_{k=1}^{L} \frac{\sinh(\lambda_k + \frac{\pi i}{12})}{\sinh(\lambda_k - \frac{\pi i}{12})}. \tag{2.11}$$

These equations have been solved to find the order one excitation energies [13]. The results are expressed in terms of three single particle excitation energies:

$$e_{2s}(P) = 3\{\sqrt{2}\cos(\frac{|P|}{2} - \frac{3\pi}{4}) + 1\}, \tag{2.12a}$$

$$e_{-2s}(P) = 3\{\sqrt{2}\cos(\frac{|P|}{2} - \frac{\pi}{4}) - 1\}, \tag{2.12b}$$

$$e_{ns}(P) = 3\sin(\frac{|P|}{2}). \tag{2.12c}$$

For $P \sim 0$, all three excitations are of the form (1.5) with

$$v = 3/2. \tag{2.13}$$

Here and in the remainder of the paper we take M to be even.

1. For $Q=0$ the energies and momenta are of the form (1.2) and (1.3)

$$E(\{P_j^{2s}\},\{P_j^{-2s}\},\{P_j^{ns}\})-E_{GS}=\sum_{\alpha=2s,-2s,ns}\sum_{j=1}^{m_\alpha}e_\alpha(P_j) \tag{2.14}$$

and

$$P=P^0+\sum_{\alpha=2s,-2s,ns}\sum_{j=1}^{m_\alpha}P_j^\alpha, \tag{2.15}$$

where

$$P^0=P_{GS}=\frac{M}{2}\pi \quad \mathrm{mod} 2\pi, \tag{2.16}$$

$$m_{2s}+m_{-2s} \quad \text{is} \quad \text{even}, \tag{2.17}$$

P_j^{2s}, P_j^{-2s}, and P_j^{ns} obey the fermi exclusion rule (1.4) and they lie in the ranges

$$0\le P_j^{2s}\le 3\pi, \tag{2.18a}$$

$$0\le P_j^{-2s}\le \pi, \tag{2.18b}$$

$$0\le P_j^{ns}\le 2\pi. \tag{2.18c}$$

We also note that the C parity of the ground state is

$$C_{GS}=(-1)^{M/2}, \tag{2.19}$$

and the C parity of an arbitrary state is

$$C/C_{GS}=(-1)^{m_{ns}+m_{-2s}+(m_{2s}+m_{-2s})/2}. \tag{2.20}$$

2. For $Q=\pm1$ we must consider $m_{2s}+m_{-2s}$ to be both even and odd. When $m_{2s}+m_{-2s}$ is even there are two spectra of the form (2.14) and (2.15). In one $P^0=P_{GS}$ and in the other $P^0=P_{GS}+\pi$. In both cases the P_j^α obey (2.18). When $m_{2s}+m_{-2s}$ is odd, there are again two spectra of the form (2.14) and (2.15). In each case $P^0=P_{GS}$. In one case P_j^α satisfies (2.18), while in the other case $-P_j^\alpha$ satisfies (2.18).

The conformal field theory predictions for the model can be obtained by noting that the 3-state Potts model is the critical D_4 model in the classification of Pasquier [11]. The central charge and the conformal dimensions of the primary fields are thus obtained by specializing the finite size computations of the A_{N+1} model of Klümper and Pearce [10] to the case $N=4$ and using an orbifold construction [21] to find the results for D_4. The general result for A_N+1 at the boundary of the I/II regime is that the central charge is

$$c=\frac{2(N-1)}{N+2}, \tag{2.21}$$

and the conformal dimensions are

$$h_m^l=\frac{\ell(\ell+2)}{4(N+2)}-\frac{m^2}{4N} \quad \text{for} \quad |m|\le\ell \tag{2.22}$$

which are the same as those of the Z_N parafermion conformal field theory of Zamolodchikov and Fateev [22]. Using the symmetry $h^{\ell}_{m}=h^{N-\ell}_{N-m}$ we find for $N=4$:

$$c=1, \tag{2.23}$$

and

$$h^0_0=0,\quad h^0_2=\frac{3}{4},\quad h^0_4=1,\quad h^2_0=\frac{1}{3},\quad h^2_2=\frac{1}{12}, \tag{2.24}$$

where the first three conformal dimensions occur in $Q=0$ and the last two in $Q=\pm1$. Moreover the modular invariant partition function is that of the D_4 parafermion model [14] (1.10) where the branching functions b^{ℓ}_m can be obtained by specializing to $N=4$ the Hecke indefinite form of Kac-Peterson [7]

$$\begin{aligned} b^{\ell}_m &= Q(q)^{-2}q^{\frac{l(l+2)}{4(N+2)}-\frac{m^2}{4N}-\frac{c}{24}} \\ &\times \Bigg[\Bigg(\sum_{s\geq0}\sum_{n\geq0}-\sum_{s<0}\sum_{n<0}\Bigg)(-1)^s q^{s(s+1)/2+(l+1)n+(l+m)s/2+(N+2)(n+s)n} \\ &+ \Bigg(\sum_{s>0}\sum_{n\geq0}-\sum_{s\leq0}\sum_{n<0}\Bigg)(-1)^s q^{s(s+1)/2+(l+1)n+(l-m)s/2+(N+2)(n+s)n}\Bigg] \end{aligned} \tag{2.25}$$

for $|m|\leq\ell$, and using the symmetries

$$b^{\ell}_m=b^{\ell}_{-m}=b^{\ell}_{m+2N}=b^{N-\ell}_{N-m} \tag{2.26}$$

otherwise. An alternative form for b^{ℓ}_m is given in [23] but for our purposes the simplest form is the specialization which only occurs for $N=4$ [8]

$$b^0_0+b^0_4=f_{3,0}/\eta,\quad b^0_0-b^0_4=g_{1,0}/\eta,\quad b^0_2=f_{3,3}/2\eta,\quad b^2_0=f_{3,2}/\eta,\quad b^2_2=f_{3,1}/\eta, \tag{2.27}$$

where

$$\eta=q^{\frac{1}{24}}Q(q), \tag{2.28}$$

and

$$f_{a,b}=\sum_{n=-\infty}^{\infty}q^{a(n+\frac{b}{2a})^2},\quad g_{a,b}=\sum_{n=-\infty}^{\infty}(-1)^n q^{a(n+\frac{b}{2a})^2}. \tag{2.29}$$

This form has a simple origin in the Gaussian model with $r=\sqrt{3/2}$, which we give in appendix A. For comparison with the expansions of subsequent sections, we list the first few terms of (2.27) as

$$q^{1/24}b^0_0=1+q^2+2q^3+4q^4+5q^5+9q^6+12q^7+19q^8+25q^9+37q^{10}+\cdots) \tag{2.30a}$$

$$q^{1/24}b^0_2=q^{3/4}(1+q+2q^2+3q^3+5q^4+7q^5+12q^6+16q^7+24q^8+33q^9+47q^{10}+\cdots) \tag{2.30b}$$

$$q^{1/24}b^0_4=q(1+q+3q^2+3q^3+6q^4+8q^5+13q^6+17q^7+27q^8+35q^9+51q^{10}+\cdots) \tag{2.30c}$$

$$q^{1/24}b^2_0=q^{1/3}(1+2q+3q^2+5q^3+8q^4+13q^5+19q^6+28q^7+41q^8+58q^9+81q^{10}+\cdots) \tag{2.30d}$$

$$q^{1/24}b^2_2=q^{1/12}(1+q+3q^2+4q^3+8q^4+11q^5+18q^6+25q^7+38q^8+52q^9+76q^{10}+\cdots) \tag{2.30e}$$

We finally note that there is an alternative way to obtain these conformal field theory predictions which utilizes W_4 algebra [24] and is related to the GKO construction $\frac{(A^{(1)}_3)_1\times(A^{(1)}_3)_1}{(A^{(1)}_3)_2}$. The branching functions have been computed from this construction in terms of three dimensional sums [25–27] which, for later comparison, we give in appendix B.

3 Branching Functions for Q=0

The partition function for the Hamiltonian (2.1) is, by definition,

$$Z=\sum_n e^{-E_n/k_BT}=e^{-E_{GS}/k_BT}\sum e^{-(E_n-E_{GS})/k_BT}, \tag{3.1}$$

where for $M\to\infty$,

$$E_{GS}=Me_0-\frac{\pi cv}{6M}+O(\frac{1}{M^2}) \tag{3.2}$$

and from [10] $c=1$. To obtain the relation with the modular invariant partition function of conformal field theory we must evaluate (3.1) in the limit $M\to\infty$, $T\to 0$ with MT fixed. We intend to carry out this evaluation by making use of the quasiparticle energy spectrum (2.14).

There are, however, two questions that must be addressed before we can do this. The first is that in order for (2.14) to specify the energy levels, the momenta P_j^α must be discretely specified on the finite lattice. The second is that the evaluation leading to (2.14) is only correct to order one as $M\to\infty$ and hence in order to agree with [10] it may be necessary to add some term of the order of $1/M$ which is independent of P_j^α but which in general will depend on m_{2s}, m_{-2s} and m_{ns}. These considerations are different for $Q=0$ and $Q=\pm1$.

We consider in this section $Q=0$. We find from the previous study [16] of the competeness of the solutions of (2.1) that for given m_{2s}, m_{-2s}, and m_{ns}

$$P_j^{2s}\quad\text{takes}\quad\frac{3M}{2}-m_{ns}-\frac{m_{2s}+m_{-2s}}{2}\quad\text{values,} \tag{3.3a}$$

$$P_j^{-2s}\quad\text{takes}\quad\frac{M}{2}-m_{ns}-\frac{m_{2s}+m_{-2s}}{2}\quad\text{values,} \tag{3.3b}$$

and

$$P_j^{ns}\quad\text{takes}\quad M-m_{ns}-m_{2s}-m_{-2s}\quad\text{values.} \tag{3.3c}$$

This will be the case if P_j^α satisfies

$$\frac{\pi}{M}(m_{ns}+\frac{m_{2s}+m_{-2s}}{2}+1)\le P_j^{2s}\le 3\pi-\frac{\pi}{M}(m_{ns}+\frac{m_{2s}+m_{-2s}}{2}+1), \tag{3.4a}$$

$$\frac{\pi}{M}(m_{ns}+\frac{m_{2s}+m_{-2s}}{2}+1)\le P_j^{-2s}\le \pi-\frac{\pi}{M}(m_{ns}+\frac{m_{2s}+m_{-2s}}{2}+1), \tag{3.4b}$$

$$\frac{\pi}{M}(m_{ns}+m_{2s}+m_{-2s}+1)\le P_j^{ns}\le 2\pi-\frac{\pi}{M}(m_{ns}+m_{2s}+m_{-2s}+1), \tag{3.4c}$$

where the spacing between allowed values for P_j^α is $2\pi/M$, and $m_{2s}+m_{-2s}$ is even. It may be verified that this choice of P_j^α exactly reproduces the correct number of momenta of table 4 of [16] for each allowed set of m_{2s}, m_{-2s} and m_{ns}.

Since $M\to\infty$ and $T\to 0$ with MT fixed, only those values of P_j^α where $e_\alpha(P)$ is small of the order $1/M$ contributes to (3.1). This occurs for

$$P_j^\alpha\sim 0\quad\text{for}\quad\alpha=2s,-2s,ns, \tag{3.5a}$$

$$P_j^{-2s}\sim\pi,\quad P_j^{ns}\sim 2\pi\quad,P_j^{2s}\sim 3\pi, \tag{3.5b}$$

where we may linearize $e_\alpha(P)$ near the endpoints (3.5) as

$$e_\alpha(P) \sim vP^\alpha \quad \text{for} \quad P^\alpha \quad \text{near} \quad \text{zero}, \tag{3.6a}$$

$$e_{-2s}(P) \sim v(\pi - P^{-2s}) \quad \text{for} \quad P^{-2s} \quad \text{near} \quad \pi, \tag{3.6b}$$

$$e_{ns}(P) \sim v(2\pi - P^{ns}) \quad \text{for} \quad P^{ns} \quad \text{near} \quad 2\pi, \tag{3.6c}$$

and

$$e_{2s}(P) \sim v(3\pi - P^{2s}) \quad \text{for} \quad P^{2s} \quad \text{near} \quad 3\pi. \tag{3.6d}$$

Thus we let m_α^ℓ be the number of P_j^α near zero and m_α^r be the number of P_j^α near the end points (3.5b). We note that

$$m_\alpha^\ell + m_\alpha^r = m_\alpha. \tag{3.7}$$

We also note that if

$$m_{2s}^r + m_{-2s}^r \quad \text{is} \quad \text{odd} \tag{3.8}$$

then from (2.15) the total momentum of the state is macroscopically shifted π from the ground state value P_{GS}. These states are expected to make oscillatory contributions to the correlation functions.

Consider first the case where all $m_\alpha^r = 0$ (which by symmetry is identical to the case $m_\alpha^\ell = 0$) and evaluate the partition function (3.1) using (2.14), (3.4), and (3.6) in the case $C/C_{GS} = 1$ where by (2.20)

$$m_{ns}^\ell + m_{-2s}^\ell + (m_{2s}^\ell + m_{-2s}^\ell)/2 \quad \text{is} \quad \text{even}. \tag{3.9}$$

We present in table 1 the terms from this construction up through order q^8 where we see that they agree with the corresponding terms from the branching function $q^{1/24}b_0^0$ of (2.27). This equality has been verified to order q^{200} and thus we conclude that this construction correctly gives the branching function b_0^0.

To obtain a formula for b_0^0 from the construction, let $P_d(m,n)$ denote the number of distinct ways that the integer n can be additively partitioned into m distinct parts. Then, modifying the usual construction of a free fermi partition function in terms of $P_d(m,n)$ to account for the momentum exclusion rule (3.4), we find

$$\begin{aligned} q^{1/24}b_0^0 \;=\; & \sum_{m_{ns},m_{2s},m_{-2s}=0}^{\infty} \; \sum_{n_{ns},n_{2s},n_{-2s}=0}^{\infty} P_d(m_{ns},n_{ns})P_d(m_{2s},n_{2s})P_d(m_{-2s},n_{-2s})q^{n_{ns}+n_{2s}+n_{-2s}} \\ & \times q^{\frac{m_{ns}}{2}(m_{ns}+m_{2s}+m_{-2s}-1)} q^{\frac{(m_{2s}+m_{-2s})}{2}(m_{ns}+\frac{m_{2s}+m_{-2s}}{2}-1)} \end{aligned} \tag{3.10}$$

where $m_{2s}+m_{-2s}$ and $m_{ns}+m_{-2s}+(m_{2s}+m_{-2s})/2$ are even and $P_d(0,0)=1$ by definition. The sums over n_α are evaluated using

$$\sum_{n=0}^{\infty} P_d(m,n)q^n = \frac{q^{m(m+1)/2}}{(q)_m}, \tag{3.11}$$

where we use the standard notation

$$(q)_m = \prod_{j=1}^{m}(1-q^j), \tag{3.12}$$

and $(q)_0 = 1$ by definition. Thus we find when $m_{2s} + m_{-2s}$ is even and (3.9) holds

$$q^{1/24} b_0^0 = \sum_{m_{ns}=0} \sum_{m_{2s}=0} \sum_{m_{-2s}=0} \frac{q^{m_{ns}(m_{ns}+1)/2}}{(q)_{m_{ns}}} \frac{q^{m_{2s}(m_{2s}+1)/2}}{(q)_{m_{2s}}} \frac{q^{m_{-2s}(m_{-2s}+1)/2}}{(q)_{m_{-2s}}}$$
$$q^{\frac{m_{ns}}{2}(m_{ns}+m_{2s}+m_{-2s}-1)} q^{\frac{(m_{2s}+m_{-2s})}{2}(m_{ns}+(m_{2s}+m_{-2s})/2-1)}. \tag{3.13}$$

We may now extend these considerations to the general case where both some $m_\alpha^r \neq 0$ and some $m_\alpha^\ell \neq 0$. In this general case we note from the work of [9,10] that the contribution to the energy from regions where (3.5a) holds and the region where (3.5b) holds are independent. Combining the above considerations we have in general the expression for the low lying energy levels in the $M \to \infty$ limit:

$$E_{ex} - E_{GS} = \sum_{\alpha=2s,-2s,ns} \left\{ \sum_{j=1}^{m_\alpha^\ell} e_\alpha(P_j^{\ell.\alpha}) + \sum_{j=1}^{m_\alpha^r} e_\alpha(P_j^{r,\alpha}) \right\}, \tag{3.14}$$

where we define $P_j^{\ell,\alpha}$ and $P_j^{r,\alpha}$ to satisfy

$$\frac{\pi}{M}(m_{ns}^\ell + \frac{m_{2s}^\ell + m_{-2s}^\ell}{2} + 1) \leq P_j^{\ell,2s}, P_j^{\ell,-2s}, \tag{3.15a}$$

$$\frac{\pi}{M}(m_{ns}^\ell + m_{2s}^\ell + m_{-2s}^\ell + 1) \leq P_j^{\ell,ns}, \tag{3.15b}$$

and

$$\frac{\pi}{M}(m_{ns}^r + \frac{m_{2s}^r + m_{-2s}^r}{2} + 1) \leq P_j^{r,2s}, P_j^{r,-2s}, \tag{3.16a}$$

$$\frac{\pi}{M}(m_{ns}^r + m_{2s}^r + m_{-2s}^r + 1) \leq P_j^{r,ns}, \tag{3.16b}$$

where again, the spacing between allowed values for $P_j^{l,r}$ is $2\pi/M$ and $e_\alpha(P) = vP$ with v given by (2.13).

In table 2 we consider the cases $m_{2s}^r = 1$, $m_{-2s}^r = m_{ns}^r = 0$ and $m_{-2s}^r = 1$, $m_{2s}^r = m_{ns}^r = 0$ and compute the contribution to made to Z to order $q^{31/4}$ of the terms in (3.14) that involve only $P_j^{\ell,\alpha}$. From (2.17) we see that

$$m_{2s}^\ell + m_{-2s}^\ell \quad \text{is} \quad \text{odd}. \tag{3.17}$$

Further, the interchange $m_{2s}^\ell \leftrightarrow m_{-2s}^\ell$ leaves (3.14) invariant and from (2.20) gives $C \leftrightarrow -C$. Thus we need only consider $m_{2s}^\ell < m_{-2s}^\ell$ and find that this construction agrees with $q^{1/24} b_2^0$ of (2.30b).

In table 3 we consider the case $m_{ns}^r = 1$, $m_{2s}^r = m_{-2s}^r = 0$ and compute to order q^8 the contribution made to Z in the channel $C/C_{GS} = 1$ of the terms in (3.14) that involve only $P_j^{\ell,\alpha}$. From (2.7) and (2.20) we find that

$$m_{2s}^\ell + m_{-2s}^\ell \quad \text{even} \quad \text{and} \quad m_{ns}^\ell + m_{-2s}^\ell + (m_{2s}^\ell + m_{-2s}^\ell)/2 \quad \text{odd}. \tag{3.18}$$

We find that this agrees with $q^{1/24} b_4^0$ of (2.30c).

These above two equalities have been verified to order q^{200}.

From these constructions we can find expressions for b_2^0 and b_4^0 as we did above for b_0^0. We thus find that $q^{1/24}b_\alpha^0$ is given by (3.13) for $\alpha = 0, 2, 4$ where

$$\begin{array}{lllllll} \text{for} & b_0^0 & m_{2s}+m_{-2s} & \text{is} & \text{even} & \text{and} & m_{ns}+m_{-2s}+\dfrac{m_{2s}+m_{-2s}}{2} \quad \text{is} \quad \text{even}, \\ \text{for} & b_2^0 & m_{2s}+m_{-2s} & \text{is} & \text{odd} & \text{and} & m_{2s} < m_{-2s}, \\ \text{for} & b_4^0 & m_{2s}+m_{-2s} & \text{is} & \text{even} & \text{and} & m_{ns}+m_{-2s}+\dfrac{m_{2s}+m_{-2s}}{2} \quad \text{is} \quad \text{odd}. \end{array} \tag{3.19}$$

We may now finally construct the complete $Q = 0$ contribution to Z by using (3.14)-(3.16) in (3.1) and summing over all m_α^r and m_α^ℓ subject only to the restriction (2.17) written in the form

$$m_{2s}^r + m_{-2s}^r + m_{2s}^\ell + m_{-2s}^\ell \quad \text{even}. \tag{3.20}$$

(We note that there is no restriction corresponding to (2.20) because both channels $C = \pm 1$ are considered in the sum.) It is easy then to see that the result consists of all the terms in (1.10) which involve b_0^0 ,b_2^0 and b_4^0 where we note: 1) that the factor of 4 in front of $b_2^0(q)b_2^0(\bar{q})$ arises because of the symmetry under $m_{2s}^r \leftrightarrow m_{-2s}^r$ and $m_{2s}^\ell \leftrightarrow m_{-2s}^\ell$, and 2) terms like $b_0^0(q)b_2^0(\bar{q})$ and $b_4^0(q)b_2^0(\bar{q})$ are excluded by (3.20).

4 Branching Functions for Q=1

The channel $Q = \pm 1$ is more complicated that the channel $Q = 0$ because as seen in section 2 the spectrum of excitations has 4 separate contributions. In [16] these contributions are distinguished by the number m_{++} of $(++)$ pairs of roots, and the number m_{-+} of $(-+)$ pairs of roots where there is the sum rule:

$$m_{2s} + 2m_{ns} + 3m_{-2s} + m_{-+} + m_{++} = M - 1. \tag{4.1}$$

We found that the three cases occurred of

$$m_{-+} - m_{++} = 1, 0, -1, \tag{4.2}$$

and when $m_{-+} = m_{++}$ the spectrum is two fold degenerate. We will thus extend the considerations of section 3 by considering these three cases separately.

4.1 $\mathbf{m_{-+} - m_{++} = -1}$

In this sector the total momentum is given by (2.15) with $P^0 = P_{GS} + \pi$ and there are $M - 1$ single particle states with $m_{ns} = 1$.

We find for all three cases (4.2) from the previous study of completeness [16] that

$$P_j^{2s} \quad \text{takes} \quad M - 1 + m_{++} + m_{-2s} \quad \text{values}, \tag{4.3a}$$

$$P_j^{-2s} \quad \text{takes} \quad m_{-2s} + m_{++} - 1 \quad \text{values}, \tag{4.3b}$$

and

$$P_j^{ns} \quad \text{takes} \quad m_{ns} + 2m_{++} + 2m_{-2s} \quad \text{values}. \tag{4.3c}$$

In this present case we use $m_{-+} = m_{++} - 1$ in (4.1) to write

$$2m_{++} = M - m_{2s} - 2m_{ns} - 3m_{-2s}, \tag{4.4}$$

and thus (4.3) reduces to

$$\frac{3M}{2} - 1 - m_{ns} - \frac{m_{2s} + m_{-2s}}{2} \quad \text{values} \quad \text{for} \quad P_j^{2s}, \tag{4.5a}$$

$$\frac{M}{2} - 1 - m_{ns} - \frac{m_{2s} + m_{-2s}}{2} \quad \text{values} \quad \text{for} \quad P_j^{-2s}, \tag{4.5b}$$

and

$$M - m_{ns} - m_{2s} - m_{-2s} \quad \text{values} \quad \text{for} \quad P_j^{ns}, \tag{4.5c}$$

where $m_{2s} + m_{-2s}$ is even.This will be the case if P_j^{α} satisfies :

$$\frac{\pi}{M}(m_{ns} + \frac{m_{2s} + m_{-2s}}{2} + 2) \leq P_j^{2s} \leq 3\pi - \frac{\pi}{M}(m_{ns} + \frac{m_{2s} + m_{-2s}}{2} + 2), \tag{4.6a}$$

$$\frac{\pi}{M}(m_{ns} + \frac{m_{2s} + m_{-2s}}{2} + 2) \leq P_j^{-2s} \leq \pi - \frac{\pi}{M}(m_{ns} + \frac{m_{2s} + m_{-2s}}{2} + 2), \tag{4.6b}$$

and

$$\frac{\pi}{M}(m_{ns} + m_{2s} + m_{-2s} + 1) \leq P_j^{ns} \leq 2\pi - \frac{\pi}{2}(m_{ns} + m_{2s} + m_{-2s} + 1), \tag{4.6c}$$

where

$$P_{j+1}^{\alpha} - P_j^{\alpha} = 2\pi/M. \tag{4.7}$$

It may be verified that this choice of P_j^{α} exactly reproduces the momenta of table 6 of [16] for each allowed set of m_{2s}, m_{-2s} and m_{ns}. Following the procedure of section 3 we compute in table 4 the contribution these states will make to the partition function where we use the linearized energies (3.6) and keep all P_j^{α} near zero.

4.2 $\mathbf{m_{-+} - m_{++} = 1}$

In this case $P^0 = P_{GS}$ and there are $M - 3$ single particle states with $m_{ns} = 1$. Furthermore we find from (4.1) that

$$2m_{++} = M - 2 - m_{2s} - 2m_{ns} - 3m_{-2s}, \tag{4.8}$$

and thus there are

$$\frac{3M}{2} - 2 - m_{ns} - \frac{m_{2s} + m_{-2s}}{2} \quad \text{values} \quad \text{for} \quad P_j^{2s}, \tag{4.9a}$$

$$\frac{M}{2} - 2 - m_{ns} - \frac{m_{2s} + m_{-2s}}{2} \quad \text{values} \quad P_j^{-2s}, \tag{4.9b}$$

and

$$M - 2 - m_{ns} - m_{2s} - m_{-2s} \quad \text{values} \quad \text{for} \quad P_j^{ns}, \tag{4.9c}$$

where again $m_{2s} + m_{-2s}$ is even. This will be satisfied if P_j^α satisfies

$$\frac{\pi}{M}(m_{ns} + \frac{m_{2s} + m_{-2s}}{2} + 3) \leq P_j^{2s} \leq 3\pi - \frac{\pi}{M}(m_{ns} + \frac{m_{2s} + m_{-2s}}{2} + 3), \tag{4.10a}$$

$$\frac{\pi}{M}(m_{ns} + \frac{m_{2s} + m_{-2s}}{2} + 3) \leq P_j^{-2s} \leq \pi - \frac{\pi}{M}(m_{ns} + \frac{m_{2s} + m_{-2s}}{2} + 3), \tag{4.10b}$$

$$\frac{\pi}{M}(m_{ns} + m_{2s} + m_{-2s} + 3) \leq P_j^{ns} \leq 2\pi - \frac{\pi}{M}(m_{ns} + m_{2s} + m_{-2s} + 3), \tag{4.10c}$$

where (4.7) holds. Again it may be verified that this choice of P_j^α exactly reproduces the momenta of table 6 of [16] for the allowed values of m_{2s}, m_{-2s} and m_{ns}. In table 5 we compute the contribution these states make to the partition function where the linearized energies (3.6) are used and all momenta are kept near zero.

4.3 $\mathbf{m_{-+} = m_{++}}$

In this case $P^0 = P_{GS}$, there are $3M - 4$ single particle states with $m_{2s} = 1$ and $M - 4$ single particle states with $m_{-2s} = 1$. Furthermore

$$2m_{++} = M - 1 - m_{2s} - 2m_{ns} - 3m_{-2s}, \tag{4.11}$$

where now $m_{2s} + m_{-2s}$ is odd. In this case there is a double degeneracy and we find from [16] that there are

$$2 \times (\frac{3M}{2} - 1 - m_{ns} - \frac{m_{2s} + m_{-2s} + 1}{2}) \quad \text{values} \quad \text{for} \quad P_j^{2s}, \tag{4.12a}$$

$$2 \times (\frac{M}{2} - 1 - m_{ns} - \frac{m_{2s} + m_{-2s} + 1}{2}) \quad \text{values} \quad \text{for} \quad P_j^{-2s}, \tag{4.12b}$$

and

$$2 \times (M - 1 - m_{ns} - m_{2s} - m_{-2s}) \quad \text{values} \quad \text{for} \quad P_j^{ns}. \tag{4.12c}$$

Now in order to get a formula for the momentum which respects (4.12) we must consider two subcases. Either P_j^α satisfy

$$\frac{\pi}{M}(m_{ns} + \frac{m_{2s} + m_{-2s} + 1}{2} + 1) \leq P_j^{2s} \leq 3\pi - \frac{\pi}{M}(m_{ns} + \frac{m_{2s} + m_{-2s} + 1}{2} + 3), \tag{4.13a}$$

$$\frac{\pi}{M}(m_{ns} + \frac{m_{2s} + m_{-2s} + 1}{2} + 1) \leq P_j^{-2s} \leq \pi - \frac{\pi}{M}(m_{ns} + \frac{m_{2s} + m_{-2s} + 1}{2} + 3), \tag{4.13b}$$

$$\frac{\pi}{M}(m_{ns} + m_{2s} + m_{-2s} + 1) \leq P_j^{ns} \leq 2\pi - \frac{\pi}{M}(m_{ns} + m_{2s} + m_{-2s} + 3), \tag{4.13c}$$

or P_j^α satisfy

$$-3\pi + \frac{\pi}{M}(m_{ns} + \frac{m_{2s} + m_{-2s} + 1}{2} + 3) \leq P_j^{2s} \leq -\frac{\pi}{M}(m_{ns} + \frac{m_{2s} + m_{-2s} + 1}{2} + 1), \tag{4.14a}$$

$$-\pi + \frac{\pi}{M}(m_{ns} + \frac{m_{2s} + m_{-2s} + 1}{2} + 3) \leq P_j^{-2s} \leq -\frac{\pi}{M}(m_{ns} + \frac{m_{2s} + m_{-2s} + 1}{2} + 1), \tag{4.14b}$$

$$-2\pi + \frac{\pi}{M}(m_{ns} + m_{2s} + m_{-2s} + 3) \leq P_j^{ns} \leq -\frac{\pi}{M}(m_{ns} + m_{2s} + m_{-2s} + 1). \qquad (4.14c)$$

However, now instead of the total momentum P being given in terms of P_j^{α} by (2.15) we must introduce a shift of order $1/M$ (which is permissible because (2.15) is only derived to order one) and write that when (4.15) holds

$$P = P^0 + \sum_{\alpha=2s,-2s,ns} \sum_{j=1}^{m_\alpha} P_j^{\alpha} + \frac{\pi}{M}(\frac{m_{2s} + m_{-2s} - 1}{2}), \qquad (4.15a)$$

and when (4.16) holds

$$P = P^0 + \sum_{\alpha=2s,-2s,ns} \sum_{j=1}^{m_\alpha} P_j^{\alpha} - \frac{\pi}{M}(\frac{m_{2s} + m_{-2s} - 1}{2}). \qquad (4.15b)$$

It may be verified that the momenta computed from these rules agrees with the momenta of table 6 of [16] for the allowed values of m_{2s}, m_{-2s} and m_{ns}.

Corresponding to this momentum shift there is an energy shift as well. Thus in table 6 we compute the contribution to the partition function of those states obtained from (4.13) with P_j^{α} near zero using the linearized energies (3.6) and subtracting the shift $\frac{\pi}{M}(\frac{m_{2s}+m_{-2s}-1}{2})$. We note that the macroscopic momentum of these states are near P_{GS}. In table 7 we compute the contribution to the partition function from those states obtained from (4.14) with P_j^{2s} near -3π, P_j^{-2s} near $-\pi$ and P_j^{ns} near -2π where we add a shift $\frac{\pi}{M}(\frac{m_{2s}+m_{-2s}-1}{2})$ to the energy. The macroscopic momentum of these states is $P_{GS} + \pi$. In both cases, due to the symmetry under $2s \leftrightarrow -2s$, only the states with $m_{2s} < m_{-2s}$ are shown.

4.4 Branching functions

We may now obtain the formulas for the two branching functions of $Q = 1$, namely b_2^0 and b_2^2 by combining together the results of the three preceding subsections with the same macroscopic momentum.

Consider first the case where the macroscopic momentum is P_{GS}. This is obtained from the $m_{-+} - m_{++} = 1$ states of table 5 and the $m_{-+} = m_{++}$ states of table 6. In table 8 we compute the sum of these two contributions and see that it is identical with the corresponding terms in $q^{-1/3}q^{1/24}b_0^2$ of (2.30d).

The other case has the macroscopic momentum of $P_{GS} + \pi$ .This is obtained from the $m_{-+} - m_{++} = -1$ states of table 4 and the $m_{-+} = m_{++}$ states of table 7. In table 9 we compute the sum of these two contributions and see that it is identical with the corresponding terms in $q^{-1/12}q^{1/24}b_2^2$ of (2.30e).

As in $Q = 0$ these identities have been verified to order q^{200}.

We may now use the above construction to obtain formulas for the branching functions b_0^2 and b_2^2 by following exactly the same procedure used in section 3. Thus we find

$$
\begin{aligned}
q^{-1/3}q^{1/24}b_0^2 \;=\; & \sum_{m_{ns}=0}^{\infty}\sum_{m_{2s}=0}^{\infty}\sum_{\substack{m_{-2s}=0\\ m_{2s}+m_{-2s}\text{ even}}}^{\infty} \frac{q^{\frac{m_{ns}(m_{ns}+1)}{2}}}{(q)_{m_{ns}}}\frac{q^{\frac{m_{2s}(m_{2s}+1)}{2}}}{(q)_{m_{2s}}}\frac{q^{\frac{m_{-2s}(m_{-2s}+1)}{2}}}{(q)_{m_{-2s}}} \\
& \qquad \times q^{\frac{m_{ns}}{2}(m_{ns}+m_{2s}+m_{-2s}+1)}q^{\frac{(m_{2s}+m_{-2s})}{2}(m_{ns}+\frac{m_{2s}+m_{-2s}}{2}+1)} \\
+ & \sum_{m_{ns}=0}^{\infty}\sum_{m_{2s}=0}^{\infty}\sum_{\substack{m_{-2s}=0\\ m_{2s}+m_{-2s}\text{ odd}}}^{\infty} \frac{q^{\frac{m_{ns}(m_{ns}+1)}{2}}}{(q)_{m_{ns}}}\frac{q^{\frac{m_{2s}(m_{2s}+1)}{2}}}{(q)_{m_{2s}}}\frac{q^{\frac{m_{-2s}(m_{-2s}+1)}{2}}}{(q)_{m_{-2s}}} \\
& \qquad \times q^{\frac{m_{ns}}{2}(m_{ns}+m_{2s}+m_{-2s}-1)}q^{\frac{(m_{2s}+m_{-2s})}{2}(m_{ns}+\frac{m_{2s}+m_{-2s}}{2})}q^{-1/4}
\end{aligned} \tag{4.16a}
$$

and

$$
\begin{aligned}
q^{-1/12}q^{1/24}b_2^2 \;=\; & \sum_{m_{ns}=0}^{\infty}\sum_{m_{2s}=0}^{\infty}\sum_{\substack{m_{-2s}=0\\ m_{2s}+m_{-2s}\text{ even}}}^{\infty} \frac{q^{\frac{m_{ns}(m_{ns}+1)}{2}}}{(q)_{m_{ns}}}\frac{q^{\frac{m_{2s}(m_{2s}+1)}{2}}}{(q)_{m_{2s}}}\frac{q^{\frac{m_{-2s}(m_{-2s}+1)}{2}}}{(q)_{m_{-2s}}} \\
& \qquad \times q^{\frac{m_{ns}}{2}(m_{ns}+m_{2s}+m_{-2s}-1)}q^{\frac{(m_{2s}+m_{-2s})}{2}(m_{ns}+\frac{m_{2s}+m_{-2s}}{2})} \\
+ & \sum_{m_{ns}=0}^{\infty}\sum_{m_{2s}=0}^{\infty}\sum_{\substack{m_{-2s}=0\\ m_{2s}+m_{-2s}\text{ odd}}}^{\infty} \frac{q^{\frac{m_{ns}(m_{ns}+1)}{2}}}{(q)_{m_{ns}}}\frac{q^{\frac{m_{2s}(m_{2s}+1)}{2}}}{(q)_{m_{2s}}}\frac{q^{\frac{m_{-2s}(m_{-2s}+1)}{2}}}{(q)_{m_{-2s}}} \\
& \qquad \times q^{\frac{m_{ns}}{2}(m_{ns}+m_{2s}+m_{-2s}+1)}q^{\frac{(m_{2s}+m_{-2s})}{2}(m_{ns}+\frac{m_{2s}+m_{-2s}}{2}+1)}q^{1/4}.
\end{aligned} \tag{4.16b}
$$

4.5 Partition function

Finally we need to compute the complete partition function in the $Q = 1$ channel. Now in each of the four cases considered above there are both left and right excitations m^r_α and m^ℓ_α. Denote the four sums obtained above for all $m^r_\alpha = 0$ by S^- for $m_{-+} - m_{++} = -1$,by S^+ for $m_{-+} - m_{++} = 1$, by S^0 for $m_{-+} = m_{++}$ with $\Delta P = 0$, and by S^π for $m_{-+} = m_{++}$ with $\Delta P = \pi$. To consider the general case with both m^r_α and m^ℓ_α both nonzero, we follow the procedure of Section 3 and consider momentum restrictions for the right and left movers separately.

Consider first $m_{-+} - m_{++} = -1$. Then because of the restriction that $m_{2s} + m_{-2s} = m^r_{2s} + m^r_{-2s} + m^\ell_{-2s} + m^\ell_{-2s}$ must be even we see that $m^r_{2s} + m^r_{-2s}$ and $m^\ell_{2s} + m^\ell_{-2s}$ must be even or odd together. If both terms are even the contribution made to the partition function is $S^-(q)S^-(\bar{q})$ and if both terms are odd and the momenta are shifted properly the contribution is $S^\pi(q)S^\pi(\bar{q})$.

The contribution from $m_{-+} - m_{++} = 1$ is similar. Again $m^r_{2s} + m^r_{-2s}$ and $m^\ell_{2s} + m^\ell_{-2s}$ must be even or odd together. The even terms contribute $S^+(q)S^+(\bar{q})$ and the odd terms contribute $S^0(q)S^0(\bar{q})$.

Finally there are the terms from $m_{++} = m_{-+}$. Here $m^r_{2s} + m^r_{-2s} + m^\ell_{2s} + m^\ell_{-2s}$ is odd, thus $m^r_{2s} + m^r_{-2s}$ is even (odd) and $m^\ell_{2s} + m^\ell_{-2s}$ is odd (even). The ends where $m^{r,\ell}_{2s} + m^{r,\ell}_{-2s}$ are odd produce the terms S^0 and S^π and if the momentum shift is properly accounted for the ends with $m^{r,\ell}_{2s} + m^{r,\ell}_{-2s}$ even produce S^+ and S^-. Thus the sector $m_{++} = m_{-+}$ produces the cross terms like $S^0(q)S^+(\bar{q})$ and $S^\pi(q)S^-(\bar{q})$ and hence the desired form $b_2^2(q)b_2^2(\bar{q}) + b_0^2(q)b_0^2(\bar{q})$ in (1.10) is obtained.

5 Discussion

There are many mathematical points of the foregoing computations which remain to be clarified such as: 1) a direct proof of the equivalence of the forms (3.13),(3.19) and (4.16) with (2.25), (2.27) and the forms of appendix B; 2) the obtaining of our results directly from the functional equations [15,20] without recourse to the study of the completeness rules [16]. However the major feature of the results of sections 3 and 4 is that they directly relate the concept of branching function with that of quasi-particle. This provides insight into the physics of the model which we will discuss here in detail.

5.1 Infrared Anomaly

All the eigenvalue spectra computed here have, up to a possible constant, the additive quasi-particle form (1.2) of

$$E_{ex} - E_{GS} = \sum_{\alpha, rules} e_\alpha(P_j), \tag{5.1}$$

where there are two important features of the rules which govern the combination of energy levels. The first is the fermi exclusion property:

$$P_i^\alpha \neq P_j^\alpha \quad \text{for} \quad i \neq j. \tag{5.2}$$

This rule in conjunction with the quasi-particle form (5.1) is often used to say that the quasi-particles are fermions.

If the momenta P_j^α were such that

$$P_j^\alpha = \frac{2\pi j}{M} \tag{5.3}$$

with j an integer (or possibly half integer) these quasi-particle energies would indeed be identical with those of a genuine free Fermi gas. However the momentum set out of which the P_j are chosen is not (5.3). Instead the momenta are subject to the restrictions (3.16), (4.6), (4.13) and (4.14). These all share the feature that there is a depletion of the number of allowed momenta near the values of P where $e_\alpha(P) = 0$, which depends on the number of quasi-particles in the state. This is an intrinsic many body effect which cannot be modeled by an effective change in some one or two body property. Because this occurs for $e(P) \sim 0$ it seems appropriate to refer to this intrinsic many body effect as an infrared anomaly. This mechanism is typical of all conformal field theories that are different from free fermions or free bosons and is what can lead to central charges being different from integer or half integer.

This manner of characterizing the phenomena underlying the behavior of systems with nontrivial branching functions has, at least on the face of it, nothing to do with integrability, Virasoro algebra, modular invariance, or any other symmetry algebra. In this respect it shares the feature of generality with Haldane's [28] definition of fractional statistics. However the definition of [28] differs from that of the infrared anomaly by relying on the finiteness of the Hilbert space. This can only be achieved by imposing an ultraviolet as well as an infrared cutoff on the problem whereas, by the very name, an infrared anomaly will exist without an ultraviolet cutoff. The essential feature of [28], however, is the abandoning of a second quantized description of the excitations in the system and this is certainly a key property of the effects described above.

5.2 Specific Heat

One of the powerful results of conformal field theory is the prediction [29,30] that the specific heat C, for a system with periodic boundary condisions, is given in terms of the central charge and the velocity of sound as $T \sim 0$ as

$$C \sim \frac{\pi k_B^2 c}{3v} T. \tag{5.4}$$

The original derivations are based on conformal invariance. We give here an argument from the point of view of the infrared anomaly.

By definition the bulk free energy per site is obtained from the partition function

$$Z = \mathrm{Tr} e^{-\frac{H}{k_B T}} \tag{5.5}$$

as

$$f = -k_B T \lim_{M\to\infty} \frac{1}{M} \ln Z, \tag{5.6}$$

where

$$T > 0 \quad \text{is} \quad \text{fixed} \quad \text{and} \quad M \to \infty. \tag{5.7}$$

Then the specific heat is

$$C = -k_B T \frac{\partial^2 f}{\partial T^2}. \tag{5.8}$$

To obtain the leading behavior as $T \to 0$ of the specific heat it is sufficient restrict attention to the lowlying (order 1 or smaller) excitations of H over the ground state. These obey the quasi particle form (5.1). Hence specific heats are commonly evaluated with formulae involving single particle levels $e(P)$. In this discussion the boundary conditions are involved only in a tacit fashion.

This argument, however, is not complete, as is apparent in the observation that any energy level with $e(P) > 0$ as $M \to \infty$ will contribute only a term exponentially small in T to the specific heat. Thus the order one excitations do not contribute to the linear term (5.4). Instead it is the levels which have the property that $\lim_{M\to\infty} e(P) = 0$ that contribute to the leading behavior.

The partition function computed in conformal field theory is in the limit

$$M \to \infty, \quad T \to 0 \quad \text{with} \quad MT \quad \text{fixed}. \tag{5.9}$$

This is not the same as the limit (5.7) which defines the specific heat. However if no additional length scale appears in the system it is expected that the behavior of the specific heat computed using the prescription (5.9) where $q = e^{-\frac{2\pi v}{Mk_B T}}$ is fixed will agree when $q \to 1$ with the $T \to 0$ behavior computed using the prescription (5.7).

The $q \to 1$ behavior of Z can be computed directly from the expression of the branching functions as infinite series in q. Since these series are a direct consequence of the infrared anomaly the influence of the many body effects which determine the momentum selection rules is apparent. These selection rules can only be seen by imposing an explicit infrared cutoff on the problem. Indeed it is known [29] that free boundary conditions and periodic boundary conditions give different amplitudes of the linear in T term of the specific heat.

The above argument has relied only on a one length scale scaling argument and has not explicitly used modular invariance. Modular invariance in this context is a mathematical property of the series expressions for the branching functions which says that if we define τ as $q = e^{2\pi i\tau}$ then for a set of branching functions $b_k(\tau)$

$$b_k(-1/\tau) = \sum M_{k,l} b_l(\tau), \tag{5.10}$$

where the functions $M_{k,l}$ are independent of τ. Thus the $q \to 1$ behavior of the branching functions is given in terms of the $q \to 0$ behavior of the branching functions. This behavior is always $q^{-\frac{c}{24}+h_k}$ where h_k is the conformal dimension and c is determined from the finite size correction to the ground state energy of (3.2). The right hand side of (5.10) will thus be dominated by the term with the smallest h_k. In particular if the smallest h_k is zero the formula (5.4) results.

This argument has now related the specific heat to the finite size correction to the ground state energy and here it is apparent by definition that boundary conditions are important. Nevertheless it is useful to keep in mind the fact that the specific heat would not be affected if we changed the finite size corrections provided that the many body effect of the infrared anomaly was not changed. In that case the modular invariance would be destroyed but the specific heat would be unchanged. This is a demonstration that the concept of infrared anomaly and modular invariance are logically distinct.

5.3 Oscillations

One of the striking features of sections 3 and 4 is the fact that the branching functions b_2^0 (with conformal dimension 3/4) and b_2^2 (with conformal dimension 1/12) are obtained with a macroscopic (order 1) momentum shift from the ground state of $\Delta P = \pi$. This results from the feature found in [13] that the energies $e_{2s}(P)$ ($e_{-2s}(P)$) vanish at 3π and π. These macroscopic momentum shifts are expected to give rise to oscillatory contributions to the correlation functions of the primary operators of b_2^0 and b_2^2. On the lattice the oscillatory term should be $(-1)^N$ where N is the separation of the operators. These microscopic oscillations are, perhaps, unusual for conformal field theories but are in fact expected if we make the observation that the anti-ferromagnetic 3-state Potts model lies deep inside the incommensurate phase of the chiral Potts model [15, 31] which is characterized by oscillatory correlations [32, 33].

5.4 Lee-Yang Edge

An interesting property of the branching functions (3.13) and (4.16) obtains if we consider only the terms where $m_{2s} = m_{-2s} = 0$ where the sums reduce to the two sums

$$S_0 = \sum_{m_{ns}=0}^{\infty} \frac{q^{m_{ns}^2}}{(q)_{m_{ns}}} \tag{5.11a}$$

and

$$S_1 = \sum_{m_{ns}=0}^{\infty} \frac{q^{m_{ns}(m_{ns}+1)}}{(q)_{m_{ns}}}. \tag{5.11b}$$

These are the famous sums of Rogers-Ramanujan [34, 36]. They become modular functions if we multiply by powers of q and consider

$$c_{1,3}(\tau) = q^{-\frac{1}{60}} S_0 \quad \text{and} \quad c_{1,1}(\tau) = q^{\frac{11}{60}} S_1, \tag{5.12}$$

which, in fact, are shown in [35] to be the two characters of the nonunitary minimal model [2] with $p = 5$ and $p' = 2$

$$c_{r,s}(q) = \frac{q^{-1/24}}{Q(q)} \sum_{n=-\infty}^{\infty} \{q^{(2npp'+rp-sp')^2/4pp'} - q^{(2npp'+rp+sp')^2/4pp'}\} \tag{5.13}$$

This model has been identified [17, 18] with the field theory that describes the behavior at the Lee-Yang edge of the Ising model [19]. Thus there is a sense in which we may say that the field theory for the Lee-Yang edge is obtained by adding a perturbation to the 3-state anti-ferromagnetic Potts chain which makes the $\pm 2s$ excitations massive without affecting the ns excitations.

It is also interesting to note that the $q \to 1$ behavior of the sums (5.11) can be calculated directly without recourse to the modular transformation (5.10) by a method that makes contact with dilogarithms. As an example we consider explicitly S_0 which we write in the form obtained directly from (3.10)

$$S_0 = \sum_{m=0} \sum_{n=0} P_d(m,n) q^n q^{\frac{m}{2}(m-1)} \tag{5.14}$$

(where the subscript ns has been dropped for simplicity). To study the limit $q \to 1$ we first use the integral representation

$$\sum_{n=0}^{\infty} P_d(m,n) q^n = \frac{1}{2\pi i} \oint \frac{dz}{z^{m+1}} \prod_{l=1}^{\infty} (1 + zq^l) \tag{5.15}$$

in the sum (5.14). We note that (5.15) vanishes if $m < 0$. Thus we extend the lower limit of the sum over m from zero to $-\infty$ and interchange the summation and integration to obtain

$$S_0 = \frac{1}{2\pi i} \oint \frac{dz}{z} \prod_{l=1}^{\infty} (1 + zq^l) \sum_{m=-\infty}^{\infty} q^{\frac{m}{2}(m-1)} z^{-m}. \tag{5.16}$$

The sum over m is expressed in terms of the Jacobi theta function [37]

$$\theta_2(v,q) = \sum_{m=-\infty}^{\infty} q^{(m-\frac{1}{2})^2} e^{i\pi(2m-1)v} \tag{5.17}$$

as

$$\sum_{m=-\infty}^{\infty} q^{\frac{m}{2}(m-1)} z^{-m} = q^{-\frac{1}{8}} z^{\frac{1}{2}} \theta_2(v, q^{\frac{1}{2}}), \tag{5.18}$$

with

$$e^{i\pi v} = z^{-\frac{1}{2}}, \tag{5.19}$$

and hence

$$S_0 = \frac{1}{2\pi i} \oint \frac{dz}{z} \prod_{l=1}^{\infty} (1 + zq^l) q^{-\frac{1}{8}} z^{\frac{1}{2}} \theta_2(v, q^{\frac{1}{2}}). \tag{5.20}$$

Then using the product representation

$$\theta_2(v, q^{\frac{1}{2}}) = q^{\frac{1}{8}} (z^{\frac{1}{2}} + z^{-\frac{1}{2}}) \prod_{n=1}^{\infty} (1-q^n)(1+q^n z)(1+q^n z^{-1}), \tag{5.21}$$

we find

$$S_0 = \exp\left\{\sum_{n=1}^{\infty} \ln(1-q^n)\right\} \frac{1}{2\pi i} \oint \frac{dz}{z}(z+1) \exp\left\{\sum_{n=1}^{\infty} \{2\ln(1+zq^n) + \ln(1+z^{-1}q^n)\}\right\}. \tag{5.22}$$

We may now study behavior of S_0 as $q \to 1$ by replacing the sums in (5.22) by integrals. Thus using the definition $q = e^{-\frac{2\pi v}{Mk_BT}}$ and setting

$$x = \frac{2\pi v}{Mk_BT} \tag{5.23}$$

we find

$$\begin{aligned} S_0 \quad \sim \quad & \exp\left\{\frac{Mk_BT}{2\pi v} \int_0^{\infty} dx \ln(1-e^{-x})\right\} \\ & \times \frac{1}{2\pi i} \oint \frac{dz}{z}(z+1) \exp\left\{\frac{Mk_BT}{2\pi v} \int_0^{\infty} dx \{2\ln(1+ze^{-x}) + \ln(1+z^{-1}e^{-x})\}\right\} \end{aligned} \tag{5.24}$$

The integral over z may now be evaluated by steepest descents. The steepest decents point occurs at the values of z that satisfy

$$\ln(1+z)^2 = \ln(1+z^{-1}), \tag{5.25}$$

and thus either $z = -1$ or

$$1 + z = z^{-1}, \tag{5.26}$$

and hence we find that the steepest descents point is

$$z = \frac{\sqrt{5}-1}{2}. \tag{5.27}$$

Thus we have

$$S_0 \sim \exp\left\{\frac{Mk_BT}{2\pi v}\int_0^\infty dx\{\ln(1-e^{-x})+2\ln(1+\frac{\sqrt{5}-1}{2}e^{-x})+\ln(1+\frac{\sqrt{5}+1}{2}e^{-x})\}\right\}, \tag{5.28}$$

which, if we define

$$t = e^{-x}, \tag{5.29}$$

and recall the definition [38] of the dilogarithm

$$\mathrm{Li}_2(z) = -\int_0^z dt\frac{\ln(1-t)}{t}, \tag{5.30}$$

may be rewritten as

$$S_0 \sim \exp\left\{-\frac{Mk_BT}{2\pi v}\{\mathrm{Li}_2(1)+2\mathrm{Li}_2(\frac{1-\sqrt{5}}{2})+\mathrm{Li}_2(-\frac{\sqrt{5}+1}{2})\}\right\}. \tag{5.31}$$

Then if we note the special values of the dilogarithm

$$\mathrm{Li}_2(1) = \frac{\pi^2}{6}, \tag{5.32a}$$

$$\mathrm{Li}_2(\frac{1-\sqrt{5}}{2}) = -\frac{\pi^2}{15}+\frac{1}{2}\ln^2(\frac{\sqrt{5}-1}{2}), \tag{5.32b}$$

$$\mathrm{Li}_2(-\frac{1+\sqrt{5}}{2}) = -\frac{\pi^2}{10}-\ln^2(\frac{\sqrt{5}+1}{2}), \tag{5.32c}$$

we obtain the result

$$S_0 \sim \exp\frac{Mk_BT\pi}{30v}. \tag{5.33}$$

The identical behavior is obtained for S_1. Thus with $Z = S_0(q)S_0(\bar{q})+S_1(q)S_1(\bar{q})$, we obtain from (5.4)-(5.8) an (effective) central charge of 2/5 which agrees with [18].

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Appendix A. Gaussian construction of the branching functions

N The form of the branching function (2.27) may be simply obtained if we note that the Z_4 parafermions with the diagonal (A_5) modular invariant partition function is known to be the $r = \sqrt{3/2}$ point on the orbifold line of $c = 1$ conformal field theories (see e.g. [39]). However, we are interested in the nondiagonal (D_4) partition function which contains the current operator of dimension (1,0) in the spectrum. This model must therefore lie on the $c = 1$ Gaussian line at the compactification radius $r = \sqrt{3/2}$. Indeed, consider the general Gaussian model partition function

$$Z(r) = \frac{1}{\eta(q)\eta(\bar{q})} \sum_{m,n=-\infty}^{\infty} q^{\Delta_{m,n}} \bar{q}^{\bar{\Delta}_{m,n}}, \tag{A.1}$$

where

$$\Delta_{m,n}(r) = \frac{1}{2}(\frac{m}{2r} + nr)^2 \quad \text{and} \quad \bar{\Delta}_{m,n}(r) = \frac{1}{2}(\frac{m}{2r} - nr)^2 \tag{A.2}$$

and set $r = \sqrt{3/2}$ to obtain

$$Z(\sqrt{\frac{3}{2}}) = \frac{1}{\eta(q)\eta(\bar{q})} \sum_{m,n=-\infty}^{\infty} q^{3(\frac{m+3n}{6})^2} \bar{q}^{3(\frac{m-3n}{6})^2}. \tag{A.3}$$

Rewriting the sum as

$$Z(\sqrt{\frac{3}{2}}) = \frac{1}{\eta(q)\eta(\bar{q})} \sum_{b=0}^{5} \sum_{\substack{m,n=-\infty \\ m+3n \equiv b \bmod 6}}^{\infty} q^{3(\frac{m+3n}{6})^2} \bar{q}^{3(\frac{m-3n}{6})^2} \tag{A.4}$$

we see that

$$Z(\sqrt{\frac{3}{2}}) = \frac{1}{\eta(q)\eta(\bar{q})} \sum_{b=0}^{5} f_{3,b}(q) f_{3,b}(\bar{q}) \tag{A.5}$$

with $f_{m,n}$ defined by (2.29). Then noting the symmetry

$$f_{3,1} = f_{3,5} \quad \text{and} \quad f_{3,2} = f_{3,4} \tag{A.6}$$

we obtain precisely the Z_{pf4} of (1.10) with the expressions of (2.27) for $b_0^0 + b_4^0$, b_2^0, b_0^2, and b_2^2.

Appendix B: Branching functions of $(\mathbf{A_3^{(1)}})_\mathbf{1} \times (\mathbf{A_3^{(1)}})_\mathbf{1} / (\mathbf{A_3^{(1)}})_\mathbf{2}$

The branching functions (2.27) can also be expressed as a three dimensional sum in the following way [25–27]. Let α_i, $i = 1, 2, 3$ be the simple roots of A_3. Then

$$(\alpha_i, \alpha_j) = c_{i,j}, \tag{B.1}$$

where c is the Cartan matrix of A_3

$$c = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{B.2}$$

Let λ_i be the fundamental weights of A_3, so that $\lambda_j = \sum_k c^{-1}_{j,k}\alpha_k$. The dominant weights $\mathbf{a}^{(k)}$ of level k are defined to be:

$$\mathbf{a}^{(k)} = \sum_{i=1}^{3} a_i\lambda_i, \quad a_i \in \mathbf{Z}, \quad a_i \geq 0, \quad \sum_{i=1}^{3} a_i \leq k. \tag{B.3}$$

Let $\mathbf{r} = \mathbf{a}^{(1)} + \rho$, $\mathbf{s} = \mathbf{a}^{(2)} + \rho$, where $\rho = \sum_i \lambda_i$. Then the following branching functions are identical with (2.27):

$$b_{\mathbf{r},\mathbf{s}} = \frac{1}{\eta^3} \sum_{\mathbf{k}\in\mathbf{Q}} \sum_{w\in W} \det(w) q^{\frac{|30\mathbf{k}-6\mathbf{r}+5w(\mathbf{s})|^2}{60}} \tag{B.4}$$

where W is the Weyl group of A_3, generated by the three simple reflections:

$$\sigma_i(\beta) = \beta - (\alpha_i,\beta)\alpha_i, \quad i = 1,2,3 \tag{B.5}$$

This group has 24 elements made up of powers of the simple reflections, and $\det(w) = \pm 1$, depending on whether the minimum number of simple reflections making up the element w is even or odd. $\mathbf{Q}$ is the root lattice of A_3, i.e. $\mathbf{k} = \sum_i m_i\alpha_i$, and the sum $\sum_{\mathbf{k}\in\mathbf{Q}}$ is thus a sum over $m_i, i = 1,2,3$, from $-\infty$ to ∞.

Equation (B.4) gives 7 unique branching functions, but only 5 appear in our model: these can be obtained by setting $\mathbf{r} = \rho$ and choosing the following $\mathbf{a}^{(2)} = \mathbf{s} - \rho$:

$$\mathbf{a}^{(2)} = \begin{cases} 0, & b_{\mathbf{r},\mathbf{s}} = b^0_0,\ h_{\mathbf{r},\mathbf{s}} = 0 \\ 2\lambda_2, & b_{\mathbf{r},\mathbf{s}} = b^0_4,\ h_{\mathbf{r},\mathbf{s}} = 1 \\ 2\lambda_1, & b_{\mathbf{r},\mathbf{s}} = b^0_2,\ h_{\mathbf{r},\mathbf{s}} = 3/4 \\ \lambda_2, & b_{\mathbf{r},\mathbf{s}} = b^2_2,\ h_{\mathbf{r},\mathbf{s}} = 1/12 \\ \lambda_1 + \lambda_3, & b_{\mathbf{r},\mathbf{s}} = b^2_0,\ h_{\mathbf{r},\mathbf{s}} = 1/3. \end{cases} \tag{B.6}$$

Here, the conformal dimension $h_{\mathbf{r},\mathbf{s}}$ is defined to be:

$$h_{\mathbf{r},\mathbf{s}} = -\frac{1}{12} + \frac{|-6\mathbf{r} + 5\mathbf{s}|^2}{60}. \tag{B.7}$$

The three dimensional sum (B.4) is to be compared with the expressions (3.13) and (4.16) of the text. We note that the sum (B.4) has a power $q^{\sum 30m_j^2}$ and thus the powers of q grow much more rapidly as a function of m_j than do (3.13) and (4.16). Note also that when the sum over W is performed (B.4) contains 24 triple sums whereas (3.13) has only one and (4.16) has two.

Table 1: The terms through order q^8 in the construction of b_0^0 from the rules of section 3. The minimum momenta $P_{min}^{ns} = \frac{\pi}{M}(m_{ns} + m_{2s} + m_{-2s} + 1)$ and $P_{ns}^{\pm 2s} = \frac{\pi}{M}(m_{ns} + \frac{m_{2s}+m_{-2s}}{2} + 1)$ are obtained from (3.15). The terms in $q^{1/24}b_0^0$ are obtained from (2.30a). Here $m_{2s} + m_{-2s}$ and $m_{ns} + m_{-2s} + \frac{m_{2s}+m_{-2s}}{2}$ are even. The macroscopic momentum shift is $\Delta P = 0$.

order	m_{ns}^{ℓ}	m_{2s}^{ℓ}	m_{-2s}^{ℓ}	P_{min}^{ns}	$P_{min}^{\pm 2s}$	$\{P^{ns}, P^{2s}, P^{-2s}\}$ (units of $\frac{\pi}{M}$)	states	$q^{1/24}b_0^0$
q^0	0	0	0	–	–	$\{0,0,0\}$	1	1
q^2	0	1	1	–	$2\pi/M$	$\{0,2,2\}$	1	1
q^3	0	1	1	–	$2\pi/M$	$\{0,4,2\}, \{0,2,4\}$	2	2
q^4	0	1	1	–	$2\pi/M$	$\{0,6,2\}, \{0,4,4\}, \{0,2,6\}$	3	
	2	0	0	$3\pi/M$	–	$\{3+5,0,0\}$	1	4
q^5	0	1	1	–	$2\pi/M$	$\{0,8,2\}, \{0,6,4\}, \{0,4,6\}$,		
						$\{0,2,8\}$	4	
	2	0	0	$3\pi/M$	–	$\{3+7,0,0\}$	1	5
q^6	0	1	1	–	$2\pi/M$	$\{0,10,2\}, \{0,8,4\}, \{0,6,6\}$,		
						$\{0,4,8\}, \{0,2,10\}$	5	
	2	0	0	$3\pi/M$	–	$\{3+9,0,0\}, \{5+7,0,0\}$	2	
	1	2	0	$4\pi/M$	$3\pi/M$	$\{4,3+5,0\}$	1	
	1	0	2	$4\pi/M$	$3\pi/M$	$\{4,0,3+5\}$	1	9
q^7	0	1	1	–	$2\pi/M$	$\{0,12,2\}, \{0,10,4\}, \{0,8,6\}$,		
						$\{0,6,8\}, \{0,4,10\}, \{0,2,12\}$	6	
	2	0	0	$3\pi/M$	–	$\{3+11,0,0\}, \{5+9,0,0\}$	2	
	1	2	0	$4\pi/M$	$3\pi/M$	$\{6,3+5,0\}, \{4,3+7\}$	2	
	1	0	2	$4\pi/M$	$3\pi/M$	$\{6,0,3+5\}, \{4,0,3+7\}$	2	12
q^8	0	1	1	–	$2\pi/M$	$\{0,14,2\}, \{0,12,4\}, \{0,10,6\}$		
						$\{0,8,8\}, \{0,6,10\}, \{0,4,12\}$		
						$\{0,2,14\}$	7	
	2	0	0	$3\pi/M$	–	$\{3+13,0,0\}, \{5+11,0,0\}$,		
						$\{7+9,0,0\}$	3	
	1	2	0	$4\pi/M$	$3\pi/M$	$\{8,3+5,0\}, \{6,3+7,0\}$,		
						$\{4,3+9,0\}, \{4,5+7,0\}$	4	
	1	0	2	$4\pi/M$	$3\pi/M$	$\{8,0,3+5\}, \{6,0,3+7\}$		
						$\{4,0,3+9\}, \{4,,0,5+7\}$	4	
	0	2	2	–	$3\pi/M$	$\{0,3+5,3+5\}$	1	19

Table 2: The terms through order $q^{31/4}$ in the construction of b_2^0 from the rules of section 3. The minimum momenta $P^{ns}_{min} = \frac{\pi}{M}(m_{ns} + m_{2s} + m_{-2s} + 1)$ and $P^{\pm 2s}_{min} = \frac{\pi}{M}(m_{ns} + \frac{m_{2s}+m_{-2s}}{2} + 1)$ are obtained from (3.15). The terms in $q^{1/24}b_2^0$ are obtained from (2.30b). Here $m_{2s} + m_{-2s}$ is odd and $m_{2s} < m_{-2s}$. The macroscopic momentum shift is $\Delta P = \pi$.

order	m^ℓ_{ns}	m^ℓ_{2s}	m^ℓ_{-2s}	P^{ns}_{min}	$P^{\pm 2s}_{min}$	$\{P^{ns}, P^{2s}, P^{-2s}\}$ (units of $\frac{\pi}{M}$)	states	$q^{1/24}b_2^0$
$q^{3/4}$	0	0	1	–	$3\pi/2M$	$\{0, 0, 3/2\}$	1	1
$q^{7/4}$	0	0	1	–	$3\pi/2M$	$\{0, 0, 7/2\}$	1	1
$q^{11/4}$	0	0	1	–	$3\pi/2M$	$\{0, 0, 11/2\}$	1	
	1	0	1	$3\pi/M$	$5\pi/2M$	$\{3, 0, 5/2\}$	1	2
$q^{15/4}$	0	0	1	–	$3\pi/2M$	$\{0, 0, 15/2\}$	1	
	1	0	1	$3\pi/M$	$5\pi/2M$	$\{3, 0, 9/2\}, \{5, 0, 5/2\}$	2	3
$q^{19/4}$	0	0	1	–	$3\pi/2M$	$\{0, 0, 19/2\}$	1	
	1	0	1	$3\pi/M$	$5\pi/2M$	$\{3, 0, 13/2\}, \{5, 0, 9/2\}, \{7, 0, 5/2\}$	3	
	0	1	2	–	$5\pi/2M$	$\{0, 5/2, 5/2 + 9/2\}$	1	5
$q^{23/4}$	0	0	1	–	$3\pi/2M$	$\{0, 0, 23/2\}$	1	
	1	0	1	$3\pi/M$	$5\pi/2M$	$\{3, 0, 17/2\}, \{5, 0, 13/2\}$		
						$\{7, 0, 9/2\}, \{9, 0, 5/2\}$	4	
	0	1	2	–	$5\pi/2M$	$\{0, 9/2, 5/2 + 9/2\}, \{0, 5/2, 5/2 + 13/2\}$	2	7
$q^{27/4}$	0	0	1	–	$3\pi/2M$	$\{0, 0, 27/2\}$	1	
	1	0	1	$3\pi/M$	$5\pi/2M$	$\{3, 0, 21/2\}, \{5, 0, 17/2\}, \{7, 0, 13/2\}$		
						$\{9, 0, 9/2\}, \{11, 0, 5/2\}$	5	
	0	1	2	–	$5\pi/2M$	$\{0, 13/2, 5/2 + 9/2\}, \{0, 9/2, 5/2 + 13/2\}$		
						$\{0, 5/2, 5/2 + 17/2\}, \{0, 5/2, 9/2 + 13/2\}$	4	
	2	0	1	$4\pi/M$	$7\pi/2M$	$\{4 + 6, 0, 7/2\}$	1	
	0	0	3	–	$5\pi/2M$	$\{0, 0, 5/2 + 9/2 + 13/2\}$	1	12
$q^{31/4}$	0	0	1	–	$3\pi/2M$	$\{0, 0, 31/2\}$	1	
	1	0	1	$3\pi/M$	$5\pi/2M$	$\{3, 0, 25/2\}, \{5, 0, 21/2\}, \{7, 0, 17/2\}$		
						$\{9, 0, 13/2\}, \{11, 0, 9/2\}, \{13, 0, 5/2\}$	6	
	0	1	2	–	$5\pi/2M$	$\{0, 5/2, 5/2 + 21/2\}, \{0, 5/2, 9/2 + 17/2\}$		
						$\{0, 9/2, 5/2 + 17/2\}, \{0, 9/2, 9/2 + 13/2\}$		
						$\{0, 13/2, 5/2 + 13/2\}, \{0, 17/2, 5/2 + 9/2\}$	6	
	2	0	1	$4\pi/M$	$7\pi/2M$	$\{4 + 8, 0, 7/2\}, \{4 + 6, 0, 11/2\}$	2	
	0	0	3	–	$5\pi/2M$	$\{0, 0, 5/2 + 9/2 + 17/2\}$	1	16

Table 3: The terms through order q^8 in the construction of b_4^0 from the rules of section 3. The minimum momenta $P^{ns}_{min} = \frac{\pi}{M}(m_{ns} + m_{2s} + m_{-2s} + 1)$ and $P^{\pm}_{min} = \frac{\pi}{M}(m_{ns} + \frac{m_{2s}+m_{-2s}}{2} + 1)$ are obtained from (3.15). The terms in $q^{1/24}b_4^0$ are obtained from (2.30c). Here $m_{2s} + m_{-2s}$ is even and $m_{ns} + m_{-2s} + \frac{m_{2s}+m_{-2s}}{2}$ is odd. The macroscopic momentum shift is $\Delta P = 0$.

order	m^{ℓ}_{ns}	m^{ℓ}_{2s}	m^{ℓ}_{-2s}	P^{ns}_{min}	$P^{\pm 2s}_{min}$	$\{P^{ns}, P^{2s}, P^{-2s}\}$ (units of $\frac{\pi}{M}$)	states	$q^{1/24}b_4^0$
q^1	1	0	0	$2\pi/M$	–	$\{2,0,0\}$	1	1
q^2	1	0	0	$2\pi/M$	–	$\{4,0,0\}$	1	1
q^3	1	0	0	$2\pi/M$	–	$\{6,0,0\}$	1	
	0	2	0	–	$2\pi/M$	$\{0,2+4,0\}$	1	
	0	0	2	–	$2\pi/M$	$\{0,0,2+4\}$	1	3
q^4	1	0	0	$2\pi/M$	–	$\{8,0,0\}$	1	
	0	2	0	–	$2\pi/M$	$\{0,2+6,0\}$	1	
	0	0	2	–	$2\pi/M$	$\{0,0,2+6\}$	1	3
q^5	1	0	0	$2\pi/M$	–	$\{10,0,0\}$	1	
	0	2	0	–	$2\pi/M$	$\{0,2+8,0\}, \{0,4+6,0\}$	2	
	0	0	2	–	$2\pi/M$	$\{0,0,2+8\}, \{0,0,4+6\}$	2	
	1	1	1	$4\pi/M$	$3\pi/M$	$\{4,3,3\}$	1	6
q^6	1	0	0	$2\pi/M$	–	$\{12,0,0\}$	1	
	0	2	0	–	$2\pi/M$	$\{0,2+10,0\}, \{0,4+8,0\}$	2	
	0	0	2	–	$2\pi/M$	$\{0,0,2+10\}, \{0,0,4+8\}$	2	
	1	1	1	$4\pi/M$	$3\pi/M$	$\{4,5,3\}, \{4,3,5\}, \{6,3,3\}$	3	8
q^7	1	0	0	$2\pi/M$	–	$\{14,0,0\}$	1	
	0	2	0	–	$2\pi/M$	$\{0,2+12,0\}, \{0,4+10,0\}$,		
						$\{0,6+8,0\}$	3	
	0	0	2	–	$2\pi/M$	$\{0,0,2+12\}, \{0,0,4+10\}$,		
						$\{0,0,6+8\}$	3	
	1	1	1	$4\pi/M$	$3\pi/M$	$\{4,7,3\}, \{4,5,5\}, \{4,3,7\}$		
						$\{6,3,5\}, \{6,5,3\}, \{8,3,3\}$	6	13
q^8	1	0	0	$2\pi/M$	–	$\{16,0,0\}$	1	
	0	2	0	–	$2\pi/M$	$\{0,2+14,0\}, \{0,4+12,0\}$,		
						$\{0,6+10,0\}$	3	
	0	0	2	–	$2\pi/M$	$\{0,0,2+14\}, \{0,0,4+12\}$,		
						$\{0,0,6+10\}$	3	
	1	1	1	$4\pi/M$	$3\pi/M$	$\{4,9,3\}, \{4,7,5\}, \{4,5,7\}$		
						$\{4,3,9\}, \{6,7,3\}, \{6,5,5\}$		
						$\{6,3,7\}, \{8,5,3\}, \{8,3,5\}$		
						$\{10,3,3\}$	10	17

Table 4: The terms through order q^8 in the sector $m_{-+} - m_{++} = -1$ constructed from the rules of section 4.1. The minimum momenta $P^{ns}_{min} = \frac{\pi}{M}(m_{ns} + m_{2s} + m_{-2s} + 1)$ and $P^{\pm 2s}_{min} = \frac{\pi}{M}(m_{ns} + \frac{m_{2s}+m_{-2s}}{2} + 2)$ are obtained from (4.6). Here $m_{2s} + m_{-2s}$ is even and the macroscopic momentum shift is $\Delta P = \pi$.

order	m^ℓ_{ns}	m^ℓ_{2s}	m^ℓ_{-2s}	P^{ns}_{min}	$P^{\pm 2s}_{min}$	$\{P^{ns}, P^{2s}, P^{-2s}\}$ (units of $\frac{\pi}{M}$)	states	total
q^0	0	0	0	–	–	$\{0,0,0\}$	1	1
q^1	1	0	0	$2\pi/M$	–	$\{2,0,0\}$	1	1
q^2	1	0	0	$2\pi/M$	–	$\{4,0,0\}$	1	1
q^3	1	0	0	$2\pi/M$	–	$\{6,0,0\}$	1	
	0	1	1	–	$3\pi/M$	$\{0,3,3\}$	1	2
q^4	1	0	0	$2\pi/M$	–	$\{8,0,0\}$	1	
	0	1	1	–	$3\pi/M$	$\{0,3,5\},\{0,5,3\}$	2	
	2	0	0	$3\pi/M$	–	$\{3+5,0,0\}$	1	
	0	2	0	–	$3\pi/M$	$\{0,3+5,0\}$	1	
	0	0	2	–	$3\pi/M$	$\{0,0,3+5\}$	1	6
q^5	1	0	0	$2\pi/M$	–	$\{10,0,0\}$	1	
	0	1	1	–	$3\pi/M$	$\{0,3,7\},\{0,5,5\},\{0,7,3\}$	3	
	2	0	0	$3\pi/M$	–	$\{3+7,0,0\}$	1	
	0	2	0	–	$3\pi/M$	$\{0,3+7,0\}$	1	
	0	0	2	–	$3\pi/M$	$\{0,0,3+7\}$	1	7
q^6	1	0	0	$2\pi/M$	–	$\{12,0,0\}$	1	
	0	1	1	–	$3\pi/M$	$\{0,3,9\},\{0,5,7\},$		
						$\{0,7,5\},\{0,9,3\}$	4	
	2	0	0	$3\pi/M$	–	$\{3+9,0,0\},\{5+7,0,0\}$	2	
	0	2	0	–	$3\pi/M$	$\{0,3+9,0\},\{0,5+7,0\}$	2	
	0	0	2	–	$3\pi/M$	$\{0,0,3+7\},\{0,0,5+7\}$	2	
	1	1	1	$4\pi/M$	$4\pi/M$	$\{4,4,4\}$	1	12
q^7	1	0	0	$2\pi/M$	–	$\{14,0,0\}$	1	
	0	1	1	–	$3\pi/M$	$\{0,3,11\},\{0,5,9\},\{0,7,7\},$		
						$\{0,9,5\},\{0,11,3\}$	5	
	2	0	0	$3\pi/M$	–	$\{3+11,0,0\},\{5+9,0,0\}$	2	
	0	2	0	–	$3\pi/M$	$\{0,3+11,0\},\{0,5+9,0\}$	2	
	0	0	2	–	$3\pi/M$	$\{0,0,3+11\},\{0,0,5+9\}$	2	
	1	1	1	$4\pi/M$	$4\pi/M$	$\{4,4,6\},\{4,6,4\},\{6,4,4\}$	3	
	1	2	0	$4\pi/M$	$4\pi/M$	$\{4,4+6,0\}$	1	
	1	0	2	$4\pi/M$	$4\pi/M$	$\{4,0,4+6\}$	1	17
q^8	1	0	0	$2\pi/M$	–	$\{16,0,0\}$	1	
	0	1	1	–	$3\pi/M$	$\{0,3,13\},\{0,5,11\},\{0,7,9\},$		
						$\{0,9,7\},\{0,11,5\},\{0,13,3\}$	6	
	2	0	0	$3\pi/M$	–	$\{3+13,0,0\},\{5+11,0,0\},\{7+9,0,0\}$	3	
	0	2	0	–	$3\pi/M$	$\{0,3+13,0\},\{0,5+11,0\},\{0,7+9,0\}$	3	
	0	0	2	–	$3\pi/M$	$\{0,0,3+13\},\{0,0,5+11\},\{0,0,7+9\}$	3	
	1	1	1	$4\pi/M$	$4\pi/M$	$\{4,4,8\},\{4,8,4\},\{8,4,4\},$		
						$\{4,6,6\},\{6,4,6\},\{6,6,4\}$	6	
	1	2	0	$4\pi/M$	$4\pi/M$	$\{4,4+8,0\},\{6,4+6,0\}$	2	
	1	0	2	$4\pi/M$	$4\pi/M$	$\{4,0,4+8\},\{6,0,4+6\}$	2	26

Table 5: The terms through order q^8 in the sector $m_{-+} - m_{++} = 1$ constructed from the rules of section 4.2. The minimum momenta $P^{ns}_{min} = \frac{\pi}{M}(m_{ns} + m_{2s} + m_{-2s} + 3)$ and $P^{\pm 2s}_{min} = \frac{\pi}{M}(m_{ns} + \frac{m_{2s}+m_{-2s}}{2} + 3)$ are obtained from (4.10). Here $m_{2s}+m_{-2s}$ is even and the macroscopic momentum shift is $\Delta P = 0$.

order	m^{ℓ}_{ns}	m^{ℓ}_{2s}	m^{ℓ}_{-2s}	P^{ns}_{min}	$P^{\pm 2s}_{min}$	$\{P^{ns}, P^{2s}, P^{-2s}\}$ (units of $\frac{\pi}{M}$)	states	total
q^0	0	0	0	–	–	$\{0,0,0\}$	1	1
q^2	1	0	0	$4\pi/M$	–	$\{4,0,0\}$	1	1
q^3	1	0	0	$4\pi/M$	–	$\{6,0,0\}$	1	1
q^4	1	0	0	$4\pi/M$	–	$\{8,0,0\}$	1	
	0	1	1	–	$4\pi/M$	$\{0,4,4\}$	1	2
q^5	1	0	0	$4\pi/M$	–	$\{10,0,0\}$	1	
	0	1	1	–	$4\pi/M$	$\{0,4,6\},\{0,6,4\}$	2	
	0	2	0	–	$4\pi/M$	$\{0,4+6,0\}$	1	
	0	0	2	–	$4\pi/M$	$\{0,0,4+6\}$	1	5
q^6	1	0	0	$4\pi/M$	–	$\{12,0,0\}$	1	
	0	1	1	–	$4\pi/M$	$\{0,4,8\},\{0,6,6\},\{0,8,4\}$	3	
	0	2	0	–	$4\pi/M$	$\{0,4+8,0\}$	1	
	0	0	2	–	$4\pi/M$	$\{0,0,4+8\}$	1	
	2	0	0	$5\pi/M$	–	$\{5+7,0,0\}$	1	7
q^7	1	0	0	$4\pi/M$	–	$\{14,0,0\}$	1	
	0	1	1	–	$4\pi/M$	$\{0,4,10\},\{0,6,8\},$		
						$\{0,8,6\},\{0,10,4\}$	4	
	0	2	0	–	$4\pi/M$	$\{0,4+10,0\},\{0,6+8,0\},$	2	
	0	0	2	–	$4\pi/M$	$\{0,0,4+10\},\{0,0,6+8\}$	2	
	2	0	0	$5\pi/M$	–	$\{5+9,0,0\}$	1	10
q^8	1	0	0	$4\pi/M$	–	$\{16,0,0\}$	1	
	0	1	1	–	$4\pi/M$	$\{0,4,12\},\{0,6,10\},\{0,8,8\},$		
						$\{0,10,6\},\{0,12,4\}$	5	
	0	2	0	–	$4\pi/M$	$\{0,4+12,0\},\{0,6+10,0\},$	2	
	0	0	2	–	$4\pi/M$	$\{0,0,4+12\},\{0,0,6+10\}$	2	
	2	0	0	$5\pi/M$	–	$\{5+11,0,0\},\{7+9,0,0\}$	2	
	1	1	1	$6\pi/M$	$5\pi/M$	$\{6,5,5\}$	1	13

Table 6: The terms through order q^8 in the sector $m_{-+} = m_{++}$ with the macroscopic momentum $\Delta P = 0$ constructed from the rules of sec, 4.3. The minimum momenta $P^{ns}_{min} = \frac{\pi}{M}(m_{ns} + m_{2s} + m_{-2s} + 1)$ and $P^{\pm 2s}_{min} = \frac{\pi}{M}(m_{ns} + \frac{m_{2s}+m_{-2s}+1}{2} + 1)$ are obtained from (4.13). Here $m_{2s} + m_{-2s}$ is odd and only $m_{2s} > m_{-2s}$ are explicitly shown.

order	m^{ℓ}_{ns}	m^{ℓ}_{2s}	m^{ℓ}_{-2s}	P^{ns}_{min}	$P^{\pm 2s}_{min}$	$\{P^{ns}, P^{2s}, P^{-2s}\}$ (units of $\frac{\pi}{M}$)	shift	states	total
q^1	0	1	0	$-$	$2\pi/M$	$\{0,2,0\}$	0	1	2
q^2	0	1	0	$-$	$2\pi/M$	$\{0,4,0\}$	0	1	2
q^3	0	1	0	$-$	$2\pi/M$	$\{0,6,0\}$	0	1	
	1	1	0	$3\pi/M$	$3\pi/M$	$\{3,3,0\}$	0	1	4
q^4	0	1	0	$-$	$2\pi/M$	$\{0,8,0\}$	0	1	
	1	1	0	$3\pi/M$	$3\pi/M$	$\{3,5,0\}, \{5,3,0\}$	0	2	6
q^5	0	1	0	$-$	$2\pi/M$	$\{0,10,0\}$	0	1	
	1	1	0	$3\pi/M$	$3\pi/M$	$\{3,7,0\}, \{5,5,0\}, \{7,3,0\}$	0	3	8
q^6	0	1	0	$-$	$2\pi/M$	$\{0,12,0\}$	0	1	
	1	1	0	$3\pi/M$	$3\pi/M$	$\{3,9,0\}, \{5,7,0\}$			
						$\{7,5,0\}, \{9,3,0\}$	0	4	
	0	2	1	$-$	$3\pi/M$	$\{0,3+5,3\}$	π/M	1	12
q^7	0	1	0	$-$	$2\pi/M$	$\{0,14,0\}$	0	1	
	1	1	0	$3\pi/M$	$3\pi/M$	$\{3,11,0\}, \{5,9,0\}$			
						$\{7,7,0\}, \{9,5,0\}, \{11,3,0\}$	0	5	
	0	2	1	$-$	$3\pi/M$	$\{0,3+7,3\}, \{0,3+5,5\}$	π/M	2	
	2	1	0	$4\pi/M$	$4\pi/M$	$\{4+6,4,0\}$	0	1	18
q^8	0	1	0	$-$	$2\pi/M$	$\{0,16,0\}$	0	1	
	1	1	0	$3\pi/M$	$3\pi/M$	$\{3,13,0\}, \{5,11,0\}, \{7,9,0\}$			
						$\{9,7,0\}, \{11,5,0\}, \{13,3,0\}$	0	6	
	0	2	1	$-$	$3\pi/M$	$\{0,3+9,3\}, \{0,5+7,3\}$			
						$\{0,3+7,5\}, \{0,3+5,7\}$	π/M	4	
	2	1	0	$4\pi/M$	$4\pi/M$	$\{4+8,4,0\}, \{4+6,6,0\}$	0	2	
	0	3	0	$-$	$3\pi/M$	$\{0,3+5+7,0\}$	π/M	1	28

Table 7: The terms through order q^8 in the sector $m_{-+} = m_{++}$ with the macroscopic momentum $\Delta P = \pi$ constructed from the rules of sec, 4.3. The minimum momenta $P^{ns}_{min} = \frac{\pi}{M}(m_{ns} + m_{2s} + m_{-2s} + 3)$ and $P^{\pm 2s}_{min} = \frac{\pi}{M}(m_{ns} + \frac{m_{2s}+m_{-2s}+1}{2} + 3)$ are obtained from (4.14). Here $m_{2s} + m_{-2s}$ is odd and only $m_{2s} > m_{-2s}$ are explicitly shown.

order	m^{ℓ}_{ns}	m^{ℓ}_{2s}	m^{ℓ}_{-2s}	P^{ns}_{min}	$P^{\pm 2s}_{min}$	$\{P^{ns}, P^{2s}, P^{-2s}\}$ (units of $\frac{\pi}{M}$)	shift	states	total
q^2	0	1	0	$-$	$4\pi/M$	$\{0,4,0\}$	0	1	2
q^3	0	1	0	$-$	$4\pi/M$	$\{0,6,0\}$	0	1	2
q^4	0	1	0	$-$	$4\pi/M$	$\{0,8,0\}$	0	1	2
q^5	0	1	0	$-$	$4\pi/M$	$\{0,10,0\}$	0	1	
	1	1	0	$5\pi/M$	$5\pi/M$	$\{5,5,0\}$	0	1	4
q^6	0	1	0	$-$	$4\pi/M$	$\{0,12,0\}$	0	1	
	1	1	0	$5\pi/M$	$5\pi/M$	$\{5,7,0\},\{7,5,0\}$	0	2	6
q^7	0	1	0	$-$	$4\pi/M$	$\{0,14,0\}$	0	1	
	1	1	0	$5\pi/M$	$5\pi/M$	$\{5,9,0\},\{7,7,0\},\{9,5,0\}$	0	3	8
q^8	0	1	0	$-$	$4\pi/M$	$\{0,16,0\}$	0	1	
	1	1	0	$5\pi/M$	$5\pi/M$	$\{5,11,0\},\{7,9,0\}$			
						$\{9,7,0\},\{11,5,0\}$	0	4	
	0	2	1	$-$	$5\pi/M$	$\{0,5+7,5\}$	$-\pi/M$	1	12

Table 8: The terms through order q^8 in the sum of $m_{-+} - m_{++} = 1$ and the $\Delta P = 0$ term of $m_{-+} = m_{++}$. These are compared with the terms in $q^{-1/3}q^{1/24}b_0^2$ of (2.30d).

order	$m_{-+} - m_{++} = 1$	$m_{-+} - m_{++} = 0$	total	$q^{-1/3}q^{1/24}b_0^2$
q^0	1	0	1	1
q^1	0	2	2	2
q^2	1	2	3	3
q^3	1	4	5	5
q^4	2	6	8	8
q^5	5	8	13	13
q^6	7	12	19	19
q^7	10	18	28	28
q^8	13	28	41	41

Table 9: The terms through order q^8 of the sum of $m_{-+} - m_{++} = -1$ and the $\Delta P = \pi$ term of $m_{-+} = m_{++}$. These are compared with the terms of $q^{-1/12}q^{1/24}b_2^2$ of (2.30e)

order	$m_{-+} - m_{++} = -1$	$m_{-+} - m_{++} = 0$	total	$q^{-1/12}q^{1/24}b_2^2$
q^0	1	0	1	1
q^1	1	0	1	1
q^2	1	2	3	3
q^3	2	2	4	4
q^4	6	2	8	8
q^5	7	4	11	11
q^6	12	6	18	18
q^7	17	8	25	25
q^8	26	12	38	38

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